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Constraints on Lorentz violation from gravitational Čerenkov radiation

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ABSTRACT

Limits on gravitational Čerenkov radiation by cosmic rays are obtained and used to constrain coefficients for Lorentz violation in the gravity sector associated with operators of even mass dimensions, including orientation-dependent effects. We use existing data from cosmic-ray telescopes to obtain conservative two-sided constraints on 80 distinct Lorentz-violating operators of dimensions four, six, and eight, along with conservative one-sided constraints on three others. Existing limits on the nine minimal operators at dimension four are improved by factors of up to a billion, while 74 of our explicit limits represent stringent first constraints on nonminimal operators. Prospects are discussed for future analyses incorporating effects of Lorentz violation in the matter sector, the role of gravitational Čerenkov radiation by high-energy photons, data from gravitational-wave observatories, the tired-light effect, and electromagnetic Čerenkov radiation by gravitons.

1. Introduction

A century after its formulation, General Relativity (GR) is established as a remarkably successful classical field theory of gravity. Extending GR into the quantum domain is widely believed to require modifications of one or more of its founding principles, and identifying experimental tests to confirm this forms an interesting challenge. One central component of GR is local Lorentz invariance, which is symmetry under local rotations and boosts. Experimental tests of this invariance have undergone a renaissance in recent years [1], following the realization that tiny observable violations of Lorentz invariance could arise naturally in some quantum theories of gravity such as strings [2]. While impressive sensitivities to a broad range of possible violations in the matter sector have been achieved, searches for Lorentz violation in the gravity sector have been less extensive. In the present work, we obtain tight constraints on a large class of potential Lorentz-violating operators in the pure-gravity sector.

Deviations from local Lorentz invariance in gravity can be studied using effective field theory, which offers a model-independent approach to describing Lorentz-violating effects arising in an underlying theory of quantum gravity [3]. Within this approach, the Lagrange density describing general Lorentz violation for pure gravity is a subset of the gravitational Standard-Model Extension (SME) consisting of the usual Einstein–Hilbert and cosmological-constant terms, along with a series of additional terms containing all possible Lorentz-violating operators. In each term, the Lorentz-violating operator is formed from gravitational-field variables and is contracted with a coefficient controlling the magnitude of the effects. The Lorentz-violating operators can be organized in a series according to increasing mass dimension d in natural units, with the corresponding coefficients having mass dimensions $4-d$. Within the pure-gravity sector of this framework, most experimental studies [4–11] and theoretical investigations [12–17] have focused on minimal operators for Lorentz violation, which have mass dimension $d=4$. Some observational consequences of nonminimal operators of dimensions $d=5,6$ are known [18], and experimental studies of nonrelativistic effects of $d=6$ operators on short-range gravity have recently been performed [19–21]. For reviews see, for example, Refs. [22–24]. Here, we obtain stringent conservative constraints on certain Lorentz-violating operators of even dimensions $d \geq 4$, following from the observation of high-energy cosmic rays and the consequent limits on vacuum gravitational Čerenkov radiation.

Electromagnetic Čerenkov radiation in ponderable media has been extensively studied since its discovery in the early 1930s [25,26]. It arises when the velocity of a massive charged particle exceeds the phase velocity of light in a medium, thereby render-

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ing the particle unstable to radiation of Čerenkov light. In the presence of Lorentz violation, the vacuum acts like a refractive medium for particles with properties controlled by the coefficients for Lorentz violation [27]. Under these circumstances, a particle traveling in a vacuum with velocity exceeding that of light can produce vacuum Čerenkov radiation, which continues until the particle loses enough energy to drop below light speed. The observation of high-energy particles of various species limits the existence of vacuum Čerenkov radiation and therefore places constraints on certain coefficients for Lorentz violation in the matter sector [28–40]. Any single coefficient constraint is normally one-sided because Čerenkov radiation is possible only for superluminal particles, which typically occurs for only one coefficient sign.

Gravitational Čerenkov radiation is an analogous effect that is hypothesized to occur when the velocity of a particle exceeds the phase velocity of gravity. In principle, this could occur in conventional GR in the presence of dust, gas, or other media, but the radiation rate is suppressed by two powers of the Newton gravitational constant G_N and hence is negligible for practical purposes [41–45]. However, in the presence of Lorentz violation, vacuum gravitational Čerenkov radiation suppressed by only one power of G_N can arise and would produce energy losses of particles traveling over astrophysical distances [46–49]. The observation of high-energy cosmic rays therefore constrains certain coefficients for Lorentz violation in the gravity sector. In this work, we use observations of the energies and celestial positions of cosmic-ray events to obtain conservative limits on a large class of coefficients in the pure-gravity sector, setting stringent first constraints on many coefficients and improving certain existing limits by factors of up to a billion.

Observable effects on photon propagation arising from Lorentz violation involving operators of arbitrary d can be classified in terms of anisotropy, dispersion, birefringence, and whether they affect vacuum propagation [50]. A comparable analysis of quadratic gravitational operators of arbitrary d reveals that a similar classification holds also in the gravity sector [51]. We focus here on nonbirefringent vacuum effects involving gravitational operators of arbitrary mass dimension, which can intuitively be viewed as certain components of a derivative-dependent effective metric $\hat{\bar{s}}^{\alpha\beta}$. We obtain wave solutions for this class of modifications to the Einstein field equations, derive expressions for the rates of vacuum gravitational Čerenkov radiation of scalars, fermions, and photons, and apply the results to extract explicit conservative constraints on coefficients for Lorentz violation for even mass dimensions $4 \le d \le 8$. Throughout this work, we use the notations and conventions of Ref. [3].

2. Lorentz-violating gravitational waves

The effective gravitational field theory containing Lorentz-violating operators of arbitrary mass dimensions [3] can be linearized to produce modified Einstein equations relevant for weak-field gravity at leading order in coefficients for Lorentz violation [12,18,51]. Our focus here is on perturbative modifications that can be written in the form

$$G_{\mu\nu} = 8\pi G_N (T_M)_{\mu\nu} + \hat{\bar{s}}^{\alpha\beta} \widetilde{R}_{\alpha\mu\beta\nu}, \tag{1}$$

where G_N is the Newton gravitational constant, $(T_M)_{\mu\nu}$ is the matter energy-momentum tensor, $\widetilde{R}_{\alpha\beta\gamma\delta} \equiv \epsilon_{\alpha\beta\kappa\lambda}\epsilon_{\gamma\delta\mu\nu}R^{\kappa\lambda\mu\nu}/4$ is the double dual of the Riemann tensor, and $G_{\mu\nu}$ is the Einstein tensor. All expressions are understood to be linearized in a flat-spacetime background with Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The operator $\hat{\bar{s}}_{\mu\nu} \equiv \hat{\bar{s}}_{\nu\mu}$ is formed as a sum of terms containing coefficients $(\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1\ldots\alpha_{d-4}}$ for Lorentz violation and even powers of derivatives,

$$\hat{\bar{s}}_{\mu\nu} \equiv \sum_d (\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1\ldots\alpha_{d-4}} \partial_{\alpha_1} \ldots \partial_{\alpha_{d-4}}, \tag{2}$$

with the sum understood to range over even values $d \ge 4$. The coefficients $(\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1\ldots\alpha_{d-4}}$ are constant and assumed to be small. The $d = 4$ coefficient $\bar{s}_{\mu\nu} \equiv (\bar{s}^{(4)})_{\mu\nu}$ appears in the minimal Lorentz-violating extension of GR and has been the subject of various experimental tests [4–11]. Nonrelativistic effects from some components of the second term $(\bar{s}^{(6)})_{\mu\nu}{}^{\alpha\beta}$ have recently been experimentally constrained as well [19–21].

The perturbative change to the field equations (1) preserves diffeomorphism symmetry even though the background coefficients $(\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1\ldots\alpha_{d-4}}$ violate it. This can be understood as a consequence of the spontaneous breaking of diffeomorphism and Lorentz symmetry [52]. As a result, the usual counting of degrees of freedom in the metric fluctuation $h_{\mu\nu}$ holds, with four auxiliary components, four gauge components, and two physical gravitational modes relevant to observable physics. The additional modes arising from the higher derivative powers appear only at high energies that lie beyond the domain of validity of effective field theory. Note that the form of Eq. (1) reveals that $\hat{\bar{s}}^{\mu\nu}$ plays the role of a derivative-dependent shift of the metric, $\eta_{\mu\nu} \to \eta_{\mu\nu} - \hat{\bar{s}}_{\mu\nu}$. The perturbation also includes Lorentz-invariant contributions, with effects governed by the trace pieces of the coefficients $(\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1\ldots\alpha_{d-4}}$. More general modifications to the Einstein equations can be countenanced and classified [3,18,51], but exploring the implications of these lies outside our present scope. We remark in passing that odd powers of derivatives in the expression (2) are excluded by the requirements of hermiticity and diffeomorphism invariance.

We seek solutions to the modified Einstein equations (1) representing perturbations of the conventional linearized gravitational waves propagating in the Minkowski vacuum. The wave solutions can readily be found at leading order in $\hat{\bar{s}}^{\mu\nu}$. The conventional Einstein equations in vacuum

$$R_{\mu\nu} = 0 \tag{3}$$

hold at zeroth order, which implies that the modified Einstein equations at first order can be written in the form

$$(\eta^{\alpha\beta} + \hat{\bar{s}}^{\alpha\beta}) R_{\alpha\mu\beta\nu} = 0. \tag{4}$$

Note that working at first order in $\hat{\bar{s}}^{\mu\nu}$ in this equation requires keeping both zeroth- and first-order contributions to the contraction of the Riemann tensor with $\eta^{\alpha\beta}$, but only zeroth-order contributions to that with $\hat{\bar{s}}^{\alpha\beta}$. To fix the gauge, we adopt the modified Hilbert condition

$$\partial_\alpha (\eta^{\alpha\beta} + \hat{\bar{s}}^{\alpha\beta}) h_{\beta\mu} = \tfrac{1}{2} \partial_\mu (\eta^{\alpha\beta} + \hat{\bar{s}}^{\alpha\beta}) h_{\alpha\beta} \tag{5}$$

and the traceless condition $(\eta^{\alpha\beta} + \hat{\bar{s}}^{\alpha\beta}) h_{\alpha\beta} = 0$ as natural choices, in light of the interpretation of the perturbation as a shift of the inverse metric.

To find the wave solutions, it is convenient to convert to momentum space via the ansatz

$$h_{\mu\nu}(x) = A_{\mu\nu}(l) e^{i l_\alpha x^\alpha}, \tag{6}$$

where l^α is the 4-momentum of the gravitational wave and where as usual only the real part of the right-hand side is taken. This implies the replacement $\partial_\alpha \to i l_\alpha$ can be adopted in the definition (2) whenever $\hat{\bar{s}}_{\mu\nu}$ acts on $h_{\mu\nu}$. For $d = 4$, the quadratic momentum dependence implies that the result tracks the conventional case modulo the deformation of the Minkowski metric. However, for $d \ge 6$ the corrections to the usual solutions involve higher powers of the 4-momentum l^α.

Using the ansatz (6), the modified Einstein equation (4) in the gauge (5) takes the form

$$(\eta^{\alpha\beta} + \hat{\hat{s}}^{\alpha\beta}) l_\alpha l_\beta A_{\mu\nu} = 0. \tag{7}$$

The resulting dispersion relation for the gravitational waves,

$$l_0^2 = \vec{l}^2 + \hat{\hat{s}}^{\alpha\beta} l_\alpha l_\beta, \tag{8}$$

suggests introducing an effective vacuum refractive index $n = n(\vec{l})$ for gravitational waves, given by

$$n^2 = 1 - \hat{\hat{s}}^{\alpha\beta} \hat{l}_\alpha \hat{l}_\beta, \tag{9}$$

where $\hat{l}_\alpha \equiv l_\alpha / l_0$. We emphasize that for $d \geq 6$ the refractive index n depends on the momentum and hence the energy of the gravitational wave and that for all d it receives direction-dependent Lorentz-violating contributions as well as both isotropic Lorentz-violating ones and Lorentz-invariant ones. The group velocity $\vec{v}_g$ can be obtained by differentiating l_0 with respect to $\vec{l}$, yielding

$$|\vec{v}_g| = 1 + \tfrac{1}{2} \sum_d (-1)^{d/2} (d-3) l^{d-4} \hat{l}^\mu \hat{l}^\nu \hat{l}_{\alpha_1} \ldots \hat{l}_{\alpha_{d-4}} (\bar{s}^{(d)})_{\mu\nu}{}^{\alpha_1 \ldots \alpha_{d-4}}. \tag{10}$$

At leading order in $\hat{\hat{s}}^{\mu\nu}$, the wave vector l_α can be written in terms of the conventional wave vector $l_\alpha^{(0)}$ in the form

$$l_\alpha = l_\alpha^{(0)} - \tfrac{1}{2} \hat{\hat{s}}_\alpha{}^\beta l_\beta^{(0)}, \tag{11}$$

where $\hat{\hat{s}}_\alpha{}^\beta = \hat{\hat{s}}_\alpha{}^\beta(l)$ is evaluated using $l^{(0)}$. The amplitude of the wave (6) can similarly be written in terms of the conventional plus and cross modes of GR. To obtain an explicit expression, it is convenient to work with trace-reversed quantities defined at first order by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \tfrac{1}{2} \eta^{(1)}_{\mu\nu} h_{\alpha\beta} \eta^{(1)\alpha\beta}, \tag{12}$$

where $\eta^{(1)}_{\mu\nu} \equiv \eta_{\mu\nu} - \hat{\hat{s}}_{\mu\nu}$ and $\eta^{(1)\mu\nu} \equiv \eta^{\mu\nu} + \hat{\hat{s}}^{\mu\nu}$. In terms of the trace-reversed amplitude $\bar{A}_{\mu\nu}$, the gauge condition (5) then takes the form

$$l_\alpha (\eta^{\alpha\beta} + \hat{\hat{s}}^{\alpha\beta}) \bar{A}_{\beta\mu} = 0. \tag{13}$$

This yields an expression for the trace-reversed amplitude $\bar{A}_{\mu\nu}$ in terms of the conventional trace-reversed amplitude $\bar{A}^{(0)}_{\mu\nu}$,

$$\bar{A}_{\mu\nu} = \bar{A}^{(0)}_{\mu\nu} - \hat{\hat{s}}_{(\mu}{}^\alpha \bar{A}^{(0)}_{\alpha\nu)}, \tag{14}$$

where the symmetrization on the indices μ, ν includes a factor of 1/2. The result also shows that the graviton polarization matrix $\epsilon_{\mu\nu}$ appearing in the matrix elements for quantum processes with gravitons takes the form

$$\epsilon_{\mu\nu} = N(\epsilon^{(0)}_{\mu\nu} - \hat{\hat{s}}_{(\mu}{}^\alpha \bar{\epsilon}^{(0)}_{\alpha\nu)}), \tag{15}$$

where $\epsilon^{(0)}_{\mu\nu}$ is the usual graviton polarization matrix and N is a normalization factor.

3. Gravitational Čerenkov radiation

A particle of any species traveling faster than the phase velocity of gravity becomes unstable to gravitational Čerenkov radiation. The differential rate to radiate a single graviton of momentum l_μ is given by

$$d\Gamma = \frac{1}{2\sqrt{m_w^2 + \vec{p}^2}} \frac{d^3k}{(2\pi)^3 2k_0} \frac{d^3l}{(2\pi)^3 2l_0} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p-k-l), \tag{16}$$

where p_μ is the incoming particle momentum obeying the dispersion relation $p_0^2 = \vec{p}^2 + m_w^2$ for a particle of species w and mass m_w, $k_\mu = p_\mu - l_\mu$ is the outgoing particle momentum, and $\mathcal{M}$ is the matrix element for the decay in the quantum field theory. The integrated rate of energy loss is therefore given by

$$\frac{dE}{dt} = -\frac{1}{4p_0} \int \frac{d^3k}{(2\pi)^3 2k_0} \frac{d^3l}{(2\pi)^3} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p-k-l). \tag{17}$$

Consider first the kinematical aspects of the rate of energy loss (17). Since the applications to follow involve $m_w \ll p_0$, we can neglect contributions of order m_w times Lorentz violation, which implies that in the delta function both $\hat{l}$ and $\hat{k}$ are aligned with $\hat{p}$ at leading order. Performing the integrals over the outgoing momenta $\vec{k}$ and manipulating the remaining delta function yields

$$\frac{dE}{dt} = -\frac{1}{8|\vec{p}|\sqrt{m_w^2 + \vec{p}^2}} \int \frac{d^3l}{(2\pi)^2 |\vec{l}|} |\mathcal{M}|^2 \delta(\cos\theta - \cos\theta_C), \tag{18}$$

where θ is the angle between $\vec{p}$ and $\vec{l}$. The generalized Čerenkov angle θ_C can be written as

$$\cos\theta_C = \frac{\sqrt{m_w^2 + \vec{p}^2}}{|\vec{p}|} \frac{1}{n(|\vec{l}|)} + \frac{|\vec{l}|}{2|\vec{p}|} \left(1 - \frac{1}{[n(|\vec{l}|)]^2}\right), \tag{19}$$

where the convenient notation $n(|\vec{l}|) \equiv n(|\vec{l}|\hat{p})$ is used. The result (19) is a function of both $|\vec{p}|$ and $|\vec{l}|$, and it encodes the Lorentz violation via the refractive index.

The delta function in the integrand of the rate (18) governs physical properties of the gravitational Čerenkov radiation. For example, in the limit $|\vec{l}| \ll |\vec{p}|$ the integrand acquires contributions only for radiation at the special Čerenkov angle $\theta = \theta_C$ given by $\cos\theta_C = 1/n\beta$, where β is the particle speed. This matches the well-known result for conventional electromagnetic Čerenkov radiation in a medium of refractive index n. The existence of a possible threshold velocity $\beta_{\rm th}$ below which no radiation occurs can also be seen. For example, in the above limit $\beta_{\rm th} = 1/n$, which again reproduces the classical result for photon Čerenkov radiation. The maximum angle of emission $\theta_{C,\max} = \cos^{-1}(1/n)$ occurs for an ultrarelativistic particle with $\beta \to 1$ in this case.

For calculational purposes, it is convenient to express the correction to the refractive index arising from operators of dimension d as

$$n \approx 1 - \tfrac{1}{2} \sum_d (-1)^{d/2} s^{(d)} |\vec{l}|^{d-4}, \tag{20}$$

where $s^{(d)}$ is a direction-dependent combination of coefficients for Lorentz violation given by

$$s^{(d)}(\hat{l}) \equiv (\bar{s}^{(d)})^{\mu\nu\alpha_1 \ldots \alpha_{d-4}} \hat{l}_\mu \hat{l}_\nu \hat{l}_{\alpha_1} \ldots \hat{l}_{\alpha_{d-4}} \tag{21}$$

and having mass dimension $4-d$. For the special case of an incoming photon or ultrarelativistic particle with $\beta \to 1$, we can use the form (20) for n to obtain expressions for the Čerenkov angle θ_C valid to leading order in Lorentz violation,

$$\cos\theta_C \approx 1 + \tfrac{1}{2} \sum_d (-1)^{d/2} s^{(d)} |\vec{l}|^{d-4} \left(1 - \frac{|\vec{l}|}{|\vec{p}|}\right),$$
$$\sin^2\theta_C \approx \sum_d (-1)^{(d+2)/2} s^{(d)} |\vec{l}|^{d-4} \left(1 - \frac{|\vec{l}|}{|\vec{p}|}\right). \tag{22}$$

The latter result reveals a cutoff at large graviton momentum $\vec{l}$ given by

$$|\vec{l}|_{\max} \approx |\vec{p}|. \tag{23}$$

The maximum momentum of a radiated graviton is thus approximately the momentum of the incoming particle, and this provides an upper cutoff to the energy-loss integral (18).

The explicit form of the integral (18) depends on the matrix element for the graviton emission, but dimensional analysis shows that its basic structure is universal in the ultrarelativistic limit of interest here. At tree level, the emission of a graviton is proportional to the Newton gravitational constant G_N. Also, since the Čerenkov process is forbidden in conventional physics, the decay rate and the energy-loss rate must be controlled by the relevant dimensionless combination $n-1$ of coefficients for Lorentz violation. The calculations of matrix elements performed below reveal that this dimensionless factor is $(n-1)^2$. Incorporating the various contributions to n from different d generates cross terms that complicate the calculation, so for definiteness and simplicity we proceed here under the assumption that only a single value of d is of interest for a given analysis. The energy-loss rate then becomes proportional to $(s^{(d)})^2$. The remaining dimensional factors must involve powers of the incoming momentum $|\vec{p}|$. The result of the integration (18) therefore takes the form

$$\frac{dE}{dt} = -F^w(d) G_N (s^{(d)})^2 |\vec{p}|^{2d-4}, \tag{24}$$

where $F^w(d)$ is a dimensionless factor depending on d and on the flavor w of the particle emitting the gravitational Čerenkov radiation. The time of flight of the particle from its initial energy E_i to a final energy E_f is then

$$t = \frac{\mathcal{F}^w(d)}{G_N (s^{(d)})^2} \left(\frac{1}{E_f^{2d-5}} - \frac{1}{E_i^{2d-5}} \right), \tag{25}$$

where $\mathcal{F}^w(d) \equiv (2d-5)/F^w(d)$ is another dimensionless factor.

To obtain explicit expressions for $F^w(d)$ and $\mathcal{F}^w(d)$, we must consider the matrix elements for specific processes. Here, we discuss in turn the cases where the incoming particle is a scalar, a photon, and a fermion.

Scalars radiating gravitons. We first consider gravitational Čerenkov radiation from a hypothetical real massive scalar minimally coupled to gravity via the Lagrange density

$$\mathcal{L} = -\tfrac{1}{2} e g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} e m_\phi^2 \phi^2, \tag{26}$$

where $e = \sqrt{|g|}$ is the vierbein determinant. The Feynman rule for the scalar-scalar-graviton vertex is $i\sqrt{16\pi G_N} C^\phi_{\mu\nu}$, where

$$C^\phi_{\mu\nu} = -p_\mu k_\nu + \tfrac{1}{2}\eta_{\mu\nu}(p^\alpha k_\alpha + m_\phi^2). \tag{27}$$

The second term has no effect in practice as it generates a trace of the graviton polarization in the matrix element and therefore vanishes for physical states. The squared matrix element for the tree-level process takes the form

$$|\mathcal{M}|^2 = 16\pi G_N C^\phi_{\mu\nu} C^\phi_{\alpha\beta} \epsilon_r^{\mu\nu} \epsilon_r^{\alpha\beta}, \tag{28}$$

where $\epsilon_r^{\alpha\beta}$ with $r = +, \times$ are the two physical graviton polarization modes contained in the matrix (15). Summing over these modes and inserting the Čerenkov angle, we find

$$|\mathcal{M}|^2 = 16\pi G_N (s^{(d)})^2 |\vec{l}|^{2d-8} \left(|\vec{p}|^4 - 2|\vec{l}||\vec{p}|^3 + |\vec{p}|^2|\vec{l}|^2 \right). \tag{29}$$

Upon integration, the dimension-dependent factor $\mathcal{F}^w(d)$ appearing in Eq. (25) is found for massive scalars $w \equiv \phi$ to be

$$\mathcal{F}^\phi(d) = \tfrac{1}{8}(d-2)(d-3). \tag{30}$$

In the special limit with only operators of mass dimension $d=4$ and only isotropic effects, the results match those obtained in Ref. [46].

Photons radiating gravitons. Next, consider gravitational Čerenkov radiation from a high-energy photon. The electromagnetic part of the Einstein–Maxwell Lagrange density is

$$\mathcal{L} = -\tfrac{1}{4} e g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}, \tag{31}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field strength. In TT gauge, the vierbein determinant contributes only at nonlinear order in the metric fluctuation and can be neglected here. Using the standard plane-wave ansatz for the photon, the Feynman rule for the photon-photon-graviton vertex is found to be $i\sqrt{16\pi G_N} C_{\mu\nu\lambda\rho}$, where

$$\begin{aligned} C_{\mu\nu\lambda\rho} = & -\tfrac{1}{2}\eta_{\mu\nu}(p_\lambda k_\rho + k_\lambda p_\rho) + \tfrac{1}{2} p_\nu (\eta_{\lambda\mu} k_\rho + \eta_{\rho\mu} k_\lambda) \\ & + \tfrac{1}{2} k_\mu (\eta_{\lambda\nu} p_\rho + \eta_{\rho\nu} p_\lambda) - \tfrac{1}{2} p_\alpha k_\alpha (\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu}). \end{aligned} \tag{32}$$

The square of the matrix element for the tree-level diagram is then

$$|\mathcal{M}|^2 = 2\pi G_N C_{\mu\nu\lambda\rho} C_{\alpha\beta\gamma\delta} \epsilon_s^\mu \epsilon_s^\alpha \epsilon_t'^\nu \epsilon_t'^\beta \epsilon_r^{\lambda\rho} \epsilon_r^{\gamma\delta}, \tag{33}$$

where the physical photon polarization vectors are ϵ_s^μ with $s = 1, 2$ and repeated indices r, s, and t indicate sums over polarizations. An extra factor of 1/2 has been incorporated as usual for the sum over incoming photon polarizations.

In evaluating the result (33), the condition $p^\mu k^\nu C_{\mu\nu\lambda\rho} = 0$ effectively implements the replacement $\epsilon_r^\mu \epsilon_r^\alpha \to \eta^{\mu\alpha}$. It is convenient to choose the 3 axis along $\vec{l}$ and the 2 axis along the component of $\vec{p}$ perpendicular to $\vec{l}$, which gives

$$p_1 = 0, \quad p_2 = -p_0 \sin\theta, \quad p_\mu l^\mu = -p_0 l_0 (1 - n\cos\theta), \tag{34}$$

and leads to the identities

$$\begin{aligned} p_\mu p_\nu (\epsilon^+)^{\mu\alpha} (\epsilon^+)^\nu{}_\alpha &= p_\mu p_\nu (\epsilon^\times)^{\mu\alpha} (\epsilon^\times)^\nu{}_\alpha = |\vec{p}|^2 \sin^2\theta, \\ (\epsilon^+)_{\mu\nu} (\epsilon^+)^{\mu\nu} &= (\epsilon^\times)_{\mu\nu} (\epsilon^\times)^{\mu\nu} = 2, \\ p_\mu p_\nu (\epsilon^+)^{\mu\nu} &= -|\vec{p}|^2 \sin^2\theta, \quad p_\mu p_\nu (\epsilon^\times)^{\mu\nu} = 0. \end{aligned} \tag{35}$$

Using these results simplifies the squared matrix element to the form

$$\begin{aligned} |\mathcal{M}|^2 = & \, 8\pi G_N (s^{(d)})^2 |\vec{l}|^{2d-8} \\ & \times \left(|\vec{l}|^4 - 4|\vec{l}|^3|\vec{p}| + 3|\vec{l}|^2|\vec{p}|^2 + 2|\vec{l}||\vec{p}|^3 - |\vec{p}|^4 \right). \end{aligned} \tag{36}$$

Integration reveals that the dimension-dependent factor $\mathcal{F}^w(d)$ in Eq. (25) is

$$\mathcal{F}^\gamma(d) = \frac{(d-1)(d-2)(d-3)(2d-3)}{4(4d^4 - 28d^3 + 65d^2 - 62d + 27)} \tag{37}$$

for photons $w \equiv \gamma$ radiating gravitons. Note that a key difference between the massive-particle and photon cases is that photons are always above threshold for gravitational Čerenkov radiation.

Fermions radiating gravitons. With the notation and conventions used in Eq. (12) of Ref. [3], the Lagrange density describing the minimal gravitational coupling of a relativistic fermion can be written as

$$\mathcal{L} = \tfrac{1}{2} i e e^{\mu}{}_{a} \overline{\psi} \gamma^{a} D_{\mu} \psi + \text{h.c.}, \tag{38}$$

where the gravitational degrees of freedom appear in the vierbein $e_{\mu}{}^{a}$ and in the covariant derivative D_μ via the spin connection. The leading contribution to the fermion-fermion-graviton vertex can be obtained by expanding the vierbein and noting that the spin connection contributes only at higher order. The Feynman rule for the vertex takes the form $i\sqrt{16\pi G_N} C^{\psi}_{\mu\nu}$, where

$$C^{\psi}_{\mu\nu} = \tfrac{1}{4}\gamma_\mu (p_\nu + k_\nu). \tag{39}$$

The squared matrix element then becomes

$$|\mathcal{M}|^2 = 8\pi G_N \sum_{\text{spins}} \left(\overline{u}(p) C^{\psi}_{\mu\nu} u(k) \epsilon_r^{\mu\nu}\right) \left(\overline{u}(k) C^{\psi}_{\alpha\beta} u(p) \epsilon_r^{\alpha\beta}\right), \tag{40}$$

where a factor of 1/2 is included for the usual average over the incoming fermion spins. Using the methodology developed for the photon case along with the standard projection

$$\sum_{\text{spins}} u(p) \otimes \overline{u}(p) = \gamma \cdot p \tag{41}$$

for a relativistic fermion, we find

$$|\mathcal{M}|^2 = 8\pi G_N (s^{(d)})^2 |\vec{p}| |\vec{l}|^{2d-8} \times \left(-|\vec{l}|^3 + 3|\vec{p}||\vec{l}|^2 - 4|\vec{p}|^2|\vec{l}| + 2|\vec{p}|^3\right). \tag{42}$$

The dimension-dependent factor $\mathcal{F}^w(d)$ in Eq. (25) resulting from the integration for $w \equiv \psi$ is

$$\mathcal{F}^{\psi}(d) = \frac{(d-2)(d-3)(2d-3)}{4(2d^2 - 7d + 9)}. \tag{43}$$

Note that this differs from the scalar result (30) due to an additional term arising from the sum over spins.

4. Constraints

According to the above results, the observation of a cosmic ray of species w arriving at the Earth with energy E_f after traveling a distance L along the direction $\hat{p}_\mu$ implies that the coefficients for Lorentz violation must satisfy the one-sided constraint

$$s^{(d)}(\hat{p}) \equiv (\bar{s}^{(d)})^{\mu\nu\alpha_1 \ldots \alpha_{d-4}} \hat{p}_\mu \hat{p}_\nu \hat{p}_{\alpha_1} \ldots \hat{p}_{\alpha_{d-4}} < \sqrt{\frac{\mathcal{F}^w(d)}{G_N E_f^{2d-5} L}}. \tag{44}$$

This reveals that high-energy particles originating at large distances offer the best bounds.

The species dependence in the bound (44) is encoded entirely in the factor $\mathcal{F}^w(d)$, which is given for scalars, photons, and fermions in Eqs. (30), (37), and (43), respectively. Note that these factors are finite for any finite d, and they vanish only for physically irrelevant values $d < 4$. Note also that they imply enhancements in radiated power for increasing particle spin. As a result, for fixed values of E_f and L, photons yield more sensitive bounds than fermions, and fermions more sensitive ones than scalars. The improved sensitivity increases with increasing d. For example, the ratio $R \equiv \mathcal{F}^{\gamma}(d) : \mathcal{F}^{\psi}(d) : \mathcal{F}^{\phi}(d)$ is $R \simeq 0.6 : 0.8 : 1$ for $d = 4$, but changes to $R \simeq 0.05 : 0.5 : 1$ for $d = 6$ and becomes $R \to d^{-2} : d^{-1}/2 : 1$ at large d.

To gain some initial intuition about the implications of the bound (44), consider the conservative scenario of a heavy nucleus traveling a distance $L \simeq 10$ Mpc $\simeq 10^{39}$ GeV^{-1} from a nearby active galactic nucleus and impacting the Earth with an observed cosmic-ray energy of about 100 EeV. Assuming the gravitational Čerenkov radiation occurs from a partonic fermion in the nucleus carrying about 10^8 GeV of the total cosmic-ray energy and taking the factor $\mathcal{F}^{\psi}(d)$ to be of order 1 for simplicity, we find constraints on combinations of coefficients for Lorentz violation of dimension $4-d$ of approximate order 10^{20-8d} GeV^{4-d}. Although only a crude estimate, this serves to reveal the quality of constraints from gravitational Čerenkov radiation. For example, bounds on some $d = 4$ Lorentz-violating operators should exceed by several orders of magnitude the various existing sensitivities, which are of order 10^{-5}–10^{-10} on dimensionless coefficients in the gravity sector [4–12]. Similarly, limits for the case $d = 6$ should reach 10^{-28} GeV^{-2} or so, representing stringent first constraints on this class of nonminimal coefficients in the gravity sector.

Repeating the above crude estimate but replacing the impinging cosmic ray with a high-energy photon reveals that gravitational Čerenkov radiation from photons generically provides weaker constraints. For example, even the observation of an ultra-high-energy gamma ray at 100 TeV would imply a sensitivity to Lorentz-violating operators of dimension d reduced by a factor of 10^{6d-15} compared to cosmic rays. This factor overwhelms any possible gain in sensitivity from greater photon propagation distances L, even for the most favorable case with $d = 4$ and for photons originating at cosmological distances. We therefore focus here on constraints from gravitational Čerenkov radiation by cosmic rays, deferring further consideration of high-energy photons to the discussion section below.

To obtain more definitive constraints, information about the direction of travel of the cosmic rays is required, in addition to their energy and distance of travel. Since cosmic rays impinge upon the Earth from many directions on the celestial sphere, it is natural to work with coefficients for Lorentz violation expressed in spherical coordinates rather than cartesian ones [50]. The combination $s^{(d)}$ of coefficients, which appears in the refractive index (20) and is given in terms of coefficients for Lorentz violation by Eq. (21), is an observer scalar and hence can be expanded in terms of spherical harmonics as

$$s^{(d)}(\hat{p}) = \sum_{jm} Y_{jm}(\hat{p}) \bar{s}^{(d)}_{jm}, \tag{45}$$

where jm are the usual angular quantum numbers, subject here to the restriction that j is even and $j \le d-2$. At each fixed d, there are $(d-1)^2$ independent spherical coefficients. The result (44) then becomes a constraint on the spherical coefficients $\bar{s}^{(d)}_{jm}$.

To obtain conservative bounds, we suppose that the cosmic ray primary is a nucleus of atomic weight N. Most high-energy cosmic rays are believed to be protons ($N = 1$), but some may be light nuclei or even heavy nuclei such as iron ($N = 56$) [53]. We also assume that any gravitational radiation is emitted by one of the fermionic partons in the nucleus, which carries a fraction r of the cosmic-ray energy $E_\oplus$ observed at the Earth. A conservative estimate is $r = 10\%$ [46]. We therefore take the energy E_f as $E_f = rE_\oplus/N \approx E_\oplus/560$, with the factor $\mathcal{F}^w(d)$ in the bound (44) identified as $\mathcal{F}^{\psi}(d)$ in Eq. (43). This conservative estimate therefore represents a reduction by a factor $(N/r)^{d-5/2} \simeq 560^{d-5/2}$ of the effective energy and hence of the bounds. The acceleration sites of cosmic rays are believed to be extragalactic, including possibly supermassive black holes in active galactic nuclei [54]. The nearest of these lies at a distance of a few Mpc, which offers a sense of the minimum value of the distance L. The distance is limited by spallation of the cosmic ray on photons in the cosmic microwave background [55]. Approximately 50% of protons and iron

Table 1
Cosmic-ray events and maximum energies used in this work.

Observatory	Events	$E_{\max}$ (EeV)	Ref.
AGASA	22	213	[57,58]
Fly's Eye	1	320	[59]
Haverah Park	13	159	[57,60]
HiRes	11	127	[61]
Pierre Auger	136	127	[62]
SUGAR	31	197	[57,63]
Telescope Array	60	162	[64]
Volcano Ranch	2	139	[57,65]
Yakutsk	23	160	[57,66]

Table 2
Conservative constraints on dimensionless coefficients $\bar{s}^{(4)}_{jm}$.

d	j	Lower bound	Coefficient	Upper bound
4	0	$-3\times10^{-14}<$	$\bar{s}^{(4)}_{00}$	
4	1	$-1\times10^{-13}<$	$\bar{s}^{(4)}_{10}$	$<7\times10^{-14}$
		$-8\times10^{-14}<$	Re $\bar{s}^{(4)}_{11}$	$<8\times10^{-14}$
		$-7\times10^{-14}<$	Im $\bar{s}^{(4)}_{11}$	$<9\times10^{-14}$
4	2	$-7\times10^{-14}<$	$\bar{s}^{(4)}_{20}$	$<1\times10^{-13}$
		$-7\times10^{-14}<$	Re $\bar{s}^{(4)}_{21}$	$<7\times10^{-14}$
		$-5\times10^{-14}<$	Im $\bar{s}^{(4)}_{21}$	$<8\times10^{-14}$
		$-6\times10^{-14}<$	Re $\bar{s}^{(4)}_{22}$	$<8\times10^{-14}$
		$-7\times10^{-14}<$	Im $\bar{s}^{(4)}_{22}$	$<7\times10^{-14}$

nuclei are believed to survive at distances of 100 Mpc, while for lighter nuclei the analogous distance is 20 Mpc [56]. For definiteness, we take $L \simeq 10$ Mpc.

Numerous collaborations have published data for the energies and angular positions of observed cosmic rays. Since higher-energy events provide greater sensitivity to coefficients for Lorentz violation, we restrict attention here to events with energies above 60 EeV. Table 1 provides some information about 299 observed events of this type. The first column lists the cosmic-ray observatory, the second shows the number of published events above 60 EeV used in this analysis, the third gives the maximum observed energy $E_{\max}$, and the final column provides the reference. To obtain numerical constraints, we adopt the modified simplex method of linear programming [67] detailed in Ref. [68] in the context of bounds on Lorentz violation from nongravitational Čerenkov radiation by neutrinos. In the present instance, the available dataset of 299 events is sufficiently large to place constraints on all coefficients for fixed $d=4$, $d=6$, and $d=8$ in turn. In principle, higher values of d could also be considered, and additional cosmic-ray data for energies below 60 EeV could be included as well.

Although the bound (44) is one sided for each cosmic ray, the dependence on the direction of travel and the plethora of data across much of the celestial sphere mean that independent two-sided constraints are implied for almost all spherical coefficients $\bar{s}^{(d)}_{jm}$ at each fixed d. The exception is the isotropic coefficient $\bar{s}^{(d)}_{00}$, which produces orientation-independent effects and hence can only be constrained on one side. For definiteness, we perform two independent analyses at each $d=4,6,8$. One assumes only the isotropic coefficient $\bar{s}^{(d)}_{00}$ is nonzero and yields a single one-sided constraint. The second assumes a purely anisotropic model allowing all the coefficients $\bar{s}^{(d)}_{jm}$ with $j\neq 0$ to be simultaneously nonzero and yields $d(d-2)$ independent two-sided constraints. Note that the spherical coefficients are complex when $m\neq 0$, so their real and imaginary parts must be treated as independent for this analysis.

Tables 2, 3, and 4 contain the resulting constraints on the spherical coefficients $\bar{s}^{(4)}_{jm}$, $\bar{s}^{(6)}_{jm}$, and $\bar{s}^{(8)}_{jm}$, respectively, obtained using cosmic-ray data and reported in the Sun-centered frame [69]. The initial two columns provide the values of d and j, while the corresponding spherical coefficients are listed in the third column. The numerical lower and upper bounds are given in the second and fourth columns, respectively, with units as specified in the table captions. For coefficients with $d=4$, the results in Table 2 represent improvements of factors of a thousand to a billion over existing maximal sensitivities obtained via direct laboratory measurements [1]. Note that the connection between the spherical coefficients $\bar{s}^{(4)}_{jm}$ and the usual cartesian ones is given by equations analogous to Eq. (130) of Ref. [40]. For coefficients with $d=6,8$, the results in Tables 3 and 4 are the first constraints in the literature. For $d=6$, they complement the constraints on other independent coefficients obtained in experiments testing short-range gravity [19–21]. The reader is reminded that the constraints in Tables 2, 3, and 4 are conservative: if the cosmic rays are assumed to be protons instead of iron nuclei and if the full proton energy is assumed available for gravitational Čerenkov radiation, then the displayed bounds for $d=4$, 5, and 6 would be sharpened by additional factors of more than 10^4, 10^9, and 10^{15}, respectively, even for the same propagation distance L.

Table 3
Conservative constraints on coefficients $\bar{s}^{(6)}_{jm}$ in GeV^{-2}.

d	j	Lower bound	Coefficient	Upper bound
6	0		$\bar{s}^{(6)}_{00}$	$<2\times10^{-31}$
6	1	$-6\times10^{-30}<$	$\bar{s}^{(6)}_{10}$	$<1\times10^{-29}$
		$-6\times10^{-30}<$	Re $\bar{s}^{(6)}_{11}$	$<7\times10^{-30}$
		$-8\times10^{-30}<$	Im $\bar{s}^{(6)}_{11}$	$<5\times10^{-30}$
6	2	$-1\times10^{-29}<$	$\bar{s}^{(6)}_{20}$	$<1\times10^{-29}$
		$-7\times10^{-30}<$	Re $\bar{s}^{(6)}_{21}$	$<7\times10^{-30}$
		$-9\times10^{-30}<$	Im $\bar{s}^{(6)}_{21}$	$<6\times10^{-30}$
		$-9\times10^{-30}<$	Re $\bar{s}^{(6)}_{22}$	$<6\times10^{-30}$
		$-8\times10^{-30}<$	Im $\bar{s}^{(6)}_{22}$	$<6\times10^{-30}$
6	3	$-1\times10^{-29}<$	$\bar{s}^{(6)}_{30}$	$<8\times10^{-30}$
		$-8\times10^{-30}<$	Re $\bar{s}^{(6)}_{31}$	$<7\times10^{-30}$
		$-6\times10^{-30}<$	Im $\bar{s}^{(6)}_{31}$	$<6\times10^{-30}$
		$-6\times10^{-30}<$	Re $\bar{s}^{(6)}_{32}$	$<6\times10^{-30}$
		$-7\times10^{-30}<$	Im $\bar{s}^{(6)}_{32}$	$<7\times10^{-30}$
		$-7\times10^{-30}<$	Re $\bar{s}^{(6)}_{33}$	$<5\times10^{-30}$
		$-7\times10^{-30}<$	Im $\bar{s}^{(6)}_{33}$	$<8\times10^{-30}$
6	4	$-1\times10^{-29}<$	$\bar{s}^{(6)}_{40}$	$<7\times10^{-30}$
		$-8\times10^{-30}<$	Re $\bar{s}^{(6)}_{41}$	$<5\times10^{-30}$
		$-8\times10^{-30}<$	Im $\bar{s}^{(6)}_{41}$	$<7\times10^{-30}$
		$-7\times10^{-30}<$	Re $\bar{s}^{(6)}_{42}$	$<6\times10^{-30}$
		$-8\times10^{-30}<$	Im $\bar{s}^{(6)}_{42}$	$<6\times10^{-30}$
		$-9\times10^{-30}<$	Re $\bar{s}^{(6)}_{43}$	$<4\times10^{-30}$
		$-6\times10^{-30}<$	Im $\bar{s}^{(6)}_{43}$	$<7\times10^{-30}$
		$-8\times10^{-30}<$	Re $\bar{s}^{(6)}_{44}$	$<9\times10^{-30}$
		$-4\times10^{-30}<$	Im $\bar{s}^{(6)}_{44}$	$<8\times10^{-30}$

5. Discussion

In this work, we have derived properties of gravitational waves in the presence of a class of Lorentz-violating operators of arbitrary d, used the results to derive energy losses from gravitational Čerenkov radiation, and performed an analysis of existing cosmic-ray observations to extract constraints on a variety of coefficients for Lorentz violation with $d=4,6,8$. With the exception of the bound on the isotropic coefficient $\bar{s}^{(4)}_{00}$, all the measurements reported here are the first constraints obtained from gravitational Čerenkov radiation on the corresponding Lorentz-violating terms,

Table 4
Conservative constraints on coefficients $\bar{s}^{(8)}_{jm}$ in GeV^{-4}.

d	j	Lower bound	Coefficient	Upper bound
8	0	$-7\times10^{-49}<$	$\bar{s}^{(8)}_{00}$	
8	1	$-1\times10^{-45}<$	$\bar{s}^{(8)}_{10}$	$<1\times10^{-45}$
		$-9\times10^{-46}<$	Re $\bar{s}^{(8)}_{11}$	$<8\times10^{-46}$
		$-9\times10^{-46}<$	Im $\bar{s}^{(8)}_{11}$	$<9\times10^{-46}$
8	2	$-9\times10^{-46}<$	$\bar{s}^{(8)}_{20}$	$<1\times10^{-45}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{21}$	$<8\times10^{-46}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{21}$	$<9\times10^{-46}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{22}$	$<9\times10^{-46}$
		$-1\times10^{-45}<$	Im $\bar{s}^{(8)}_{22}$	$<9\times10^{-46}$
8	3	$-1\times10^{-45}<$	$\bar{s}^{(8)}_{30}$	$<1\times10^{-45}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{31}$	$<8\times10^{-46}$
		$-9\times10^{-46}<$	Im $\bar{s}^{(8)}_{31}$	$<9\times10^{-46}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{32}$	$<9\times10^{-46}$
		$-9\times10^{-46}<$	Im $\bar{s}^{(8)}_{32}$	$<8\times10^{-46}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{33}$	$<1\times10^{-45}$
		$-1\times10^{-45}<$	Im $\bar{s}^{(8)}_{33}$	$<1\times10^{-45}$
8	4	$-1\times10^{-45}<$	$\bar{s}^{(8)}_{40}$	$<1\times10^{-45}$
		$-6\times10^{-46}<$	Re $\bar{s}^{(8)}_{41}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{41}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{42}$	$<1\times10^{-45}$
		$-6\times10^{-46}<$	Im $\bar{s}^{(8)}_{42}$	$<1\times10^{-45}$
		$-7\times10^{-46}<$	Re $\bar{s}^{(8)}_{43}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{43}$	$<8\times10^{-46}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{44}$	$<8\times10^{-46}$
		$-9\times10^{-46}<$	Im $\bar{s}^{(8)}_{44}$	$<6\times10^{-46}$
8	5	$-1\times10^{-45}<$	$\bar{s}^{(8)}_{50}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{51}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{51}$	$<7\times10^{-46}$
		$-9\times10^{-46}<$	Re $\bar{s}^{(8)}_{52}$	$<9\times10^{-46}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{52}$	$<8\times10^{-46}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{53}$	$<7\times10^{-46}$
		$-6\times10^{-46}<$	Im $\bar{s}^{(8)}_{53}$	$<1\times10^{-45}$
		$-9\times10^{-46}<$	Re $\bar{s}^{(8)}_{54}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{54}$	$<8\times10^{-46}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{55}$	$<1\times10^{-45}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{55}$	$<1\times10^{-45}$
8	6	$-1\times10^{-45}<$	$\bar{s}^{(8)}_{60}$	$<2\times10^{-45}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{61}$	$<1\times10^{-45}$
		$-7\times10^{-46}<$	Im $\bar{s}^{(8)}_{61}$	$<9\times10^{-46}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{62}$	$<6\times10^{-46}$
		$-6\times10^{-46}<$	Im $\bar{s}^{(8)}_{62}$	$<1\times10^{-45}$
		$-7\times10^{-46}<$	Re $\bar{s}^{(8)}_{63}$	$<1\times10^{-45}$
		$-7\times10^{-46}<$	Im $\bar{s}^{(8)}_{63}$	$<8\times10^{-46}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{64}$	$<1\times10^{-45}$
		$-9\times10^{-46}<$	Im $\bar{s}^{(8)}_{64}$	$<8\times10^{-46}$
		$-8\times10^{-46}<$	Re $\bar{s}^{(8)}_{65}$	$<9\times10^{-46}$
		$-8\times10^{-46}<$	Im $\bar{s}^{(8)}_{65}$	$<9\times10^{-46}$
		$-1\times10^{-45}<$	Re $\bar{s}^{(8)}_{66}$	$<9\times10^{-46}$
		$-7\times10^{-46}<$	Im $\bar{s}^{(8)}_{66}$	$<1\times10^{-45}$

and none of the coefficients for $d=6$ or 8 have previously been constrained in the literature.

The constraints in Tables 2, 3, and 4 are obtained using cosmic rays rather than photons because the former are observed at much higher energies than the latter. Nonetheless, the absence of gravitational Čerenkov radiation from high-energy photons does in principle contain additional information. Indeed, in a more general treatment, each species w would provide distinct constraints because the particle itself experiences Lorentz violation that is flavor dependent [27].

To illustrate this, consider the Lorentz-violating vacuum dispersion relation

$$n_w^2 p_0^2 - \vec{p}^{\,2} - m_w^2 = 0, \tag{46}$$

where $n_w = n_w(\vec{p})$ is a refractive index for the particle of species w, with $m_w = 0$ if the particle is a photon. The motion of the particle then follows a geodesic in a pseudo-Finsler spacetime [70,71]. An example would be a refractive index for a massive fermion given by [40,72]

$$n_w^2 = 1 + 2\hat{c}_w^{\alpha\beta}\hat{p}_\alpha\hat{p}_\beta, \tag{47}$$

in analogy with the refractive index (9) for gravity, where $\hat{c}_w^{\alpha\beta}$ is a momentum-dependent coefficient having expansion of the form (2). For $d=4$ this reduces to a special case of the matter sector of the minimal SME with a single fermion flavor [27], with $\hat{c}_w^{\alpha\beta} = c_w^{\alpha\beta}$ being a constant dimensionless coefficient for Lorentz violation. A similar result holds for neutrinos [39] and for photons [50], where the corresponding constant dimensionless coefficient is conventionally denoted by $(\overline{k_F})^{\alpha\beta}$. Repeating the analysis of gravitational Čerenkov radiation generates an energy loss given by Eq. (18) as before, but with the vacuum Čerenkov angle θ_C given instead by

$$\begin{aligned}\cos\theta_C = {} & \frac{\sqrt{m_w^2+\vec{p}^{\,2}}}{|\vec{p}|}\frac{[n_w(|\vec{p}|-|\vec{l}|)]^2}{n_w(|\vec{p}|)n(|\vec{l}|)} \\ & + \frac{|\vec{l}|}{2|\vec{p}|}\left(1-\frac{[n_w(|\vec{p}|-|\vec{l}|)]^2}{[n(|\vec{l}|)]^2}\right) \\ & + \frac{m_w^2+\vec{p}^{\,2}}{2|\vec{l}||\vec{p}|}\left(1-\frac{[n_w(|\vec{p}|-|\vec{l}|)]^2}{[n_w(|\vec{p}|)]^2}\right),\end{aligned} \tag{48}$$

where as before the arguments of the refractive index are understood to be oriented along $\hat{p}$. This expression reveals that the presence and rate of gravitational Čerenkov radiation depends on ratios of the refractive indices for the particle and the gravitational waves. Depending on the relative magnitudes of the coefficients for Lorentz violation for the particle and the graviton, gravitational Čerenkov radiation may occur in a given direction of travel for only one sign of the correction $s^{(d)}$ to the graviton refractive index, for either sign, or not at all. Note that the final term depends only on the particle refractive index, contributing only when n_w depends on momentum and hence only for $d>4$, implying the particle is experiencing nonminimal Lorentz violation. Note also that the above expression reduces to the result (19) in the limit $n_w \to 1$, as expected.

Inserting the gravitational Čerenkov angle (48) into the integral (18) for the energy loss and limiting attention to fixed d must by dimensional arguments produce a result of the general form (24) but now involving a linear combination of the quadratic terms $(s^{(d)})^2$, $s^{(d)}(1-n_w)$, and $(1-n_w)^2$, where $1-n_w$ is given in terms of matter coefficients for Lorentz violation by an expression analogous to that for $s^{(d)}$ in Eq. (21). It follows that a detailed analysis of observations of high-energy particles, including photons and neutrinos, would yield constraints on distinct combinations of coefficients for Lorentz violation. However, photons yield weaker constraints than those from cosmic rays because observed cosmic-ray energies exceed the highest photon energies by about a millionfold, which becomes scaled by the power $d-5/2$ in extracting a bound. A similar conclusion holds for neutrinos. Note that any observed but unexplained absence of ultra-high-energy particles such as neutrinos or photons could in principle be attributed to Lorentz-violating vacuum Čerenkov radiation, including the gravitational Čerenkov radiation considered here. A complete analysis along these various lines would be of interest but lies beyond our present scope.

The expression (48) for θ_C also reveals that if the Lorentz violation is minimal, so that all Lorentz-violating operators have mass dimension $d=4$ and both n and n_w are momentum independent, then the properties of the gravitational Čerenkov radiation are determined by differences of SME coefficients for Lorentz violation in the gravity and matter sectors. This leads to an interesting relation between distinct measurements, as follows. In this scenario, the rate of energy loss for a massive fermion undergoing gravitational Čerenkov radiation is governed by the combination $\bar{s}^{\alpha\beta}+2\bar{c}_w^{\alpha\beta}$, while that for a radiating photon is governed by $\bar{s}^{\alpha\beta}-(\overline{k_F})^{\alpha\beta}$. Moreover, due to the freedom to choose coordinates without changing the physics, all nongravitational searches for Lorentz violation involving these species must involve the combination $2\bar{c}_w^{\alpha\beta}+(\overline{k_F})^{\alpha\beta}$ [73]. As a consequence, if analyses yield measurements M_1 of $\bar{s}^{\alpha\beta}+2\bar{c}_w^{\alpha\beta}$, M_2 of $\bar{s}^{\alpha\beta}-(\overline{k_F})^{\alpha\beta}$, and M_3 of $2\bar{c}_w^{\alpha\beta}+(\overline{k_F})^{\alpha\beta}$, then the three measurements must satisfy the condition

$$M_1-M_2-M_3=0. \tag{49}$$

A relation of this type, relevant for searches for CPT violation with neutral-meson oscillations, has previously been inferred among SME coefficients in the quark sector [74].

The components of the coefficients $(\bar{s}^{(d)})^{\mu\nu\alpha_1\ldots\alpha_{d-4}}$ constrained in this work can also be measured in other ways. For example, all the corresponding Lorentz-violating operators for $d>4$ are dispersive and so can in principle be measured using pulse-spread data from a terrestrial gravitational-wave observatory such as the Advanced Laser Interferometer Gravitational-Wave Observatory (LIGO) [75] and Advanced Virgo [76] or from a space observatory such as the proposed Evolved Laser Interferometer Space Antenna (eLISA) [77], assuming gravitational waves are indeed detected. The sensitivity of dispersion measurements of $s^{(d)}$ depends on the ratio of the observed pulse width to the source distance L, which typically leads to weaker constraints than those from gravitational Čerenkov radiation. However, dispersion limits are distinct and unique in detail because they involve Lorentz violation in the electron and photon sectors by virtue of the laser interactions with the mirrors. Moreover, dispersive measurements involve the group velocity and hence are sensitive to the gradient of the refractive index instead of the refractive index itself, so the corresponding measurements constrain different combinations of Lorentz-violating operators when more than one value of d is incorporated.

Another interesting open issue is the prospects for independent two-sided bounds on the isotropic coefficients $\bar{s}_{00}^{(d)}$, which cannot be obtained via gravitational Čerenkov radiation. Gravitational-wave observatories such as LIGO and eLISA are uniquely suited to place dispersion limits on these coefficients for operators of dimension $d>4$. However, no dispersion occurs for the minimal isotropic coefficient $\bar{s}_{00}^{(4)}$, which makes its two-sided measurement challenging. One possibility in principle would be to compare the time of flight of gravitational waves to that of light or neutrinos emitted from the same source. This has the disadvantage of requiring simultaneous observation using different techniques. Other options already used to place constraints on $\bar{s}_{00}^{(4)}$ include analyses of data from Gravity Probe B [9] and from pulsar timing [10]. Methods such as the study of orbital decay rates of binary systems [78] have the potential to provide interesting sensitivities to $\bar{s}_{00}^{(4)}$ as well.

The derivations in this work may also have implications for other ideas. For example, the presence of Lorentz violation in quantum electrodynamics can induce triple photon splitting in the vacuum and hence reduce the frequency of light as a function of distance traveled, suggesting the possibility of modifications to the usual interpretation of the observed cosmological redshift [79]. A similar possibility is implied by the expressions (24), (25), and (37) for the energy loss of a photon due to gravitational Čerenkov radiation. These ideas are conceptually akin to 'tired-light' models [80], which are strongly constrained by the direct observation of time dilation associated with cosmological redshift [81]. However, the energy losses (24) here have distinctive frequency dependence, and in principle they might only be perturbative or only affect part of the observed redshifts, perhaps such as those associated with supernova studies of dark energy. A complete discussion of these possibilities would require analysis of gravitational-wave propagation in a cosmological background instead of the static Minkowski background adopted here. Nonetheless, the analysis in the present work suffices to provide simple estimates of the possible scale of the effects, as follows.

For an astrophysical source at small redshift z defined in terms of the photon energies as usual by $z+1=E_i/E_f$, the luminosity distance L_L can be written as $L_\mathrm{L}\approx(z+O(z^2))/H_0$, where $H_0\simeq 1.5\times10^{-42}$ GeV is the Hubble constant [82]. Directly expressing the time of flight (25) as a distance L in terms of z gives

$$L\approx\frac{\mathcal{F}^w(d)}{G_N(s^{(d)})^2E_i^{2d-5}}\left((z+1)^{2d-5}-1\right). \tag{50}$$

Comparing L_L and L reveals that the potential contribution of gravitational Čerenkov radiation to the observed cosmological redshift is primarily governed by the dimensionless ratio

$$R\equiv\frac{G_N(s^{(d)})^2E_i^{2d-5}}{H_0}\simeq10^4E_i^3(n-1)^2, \tag{51}$$

where E_i is measured in GeV and values $R\gtrsim1$ represent substantial effects. This suggests that redshift modifications from gravitational Čerenkov radiation are negligible for most practical purposes. For example, for the optical frequencies $\simeq$ 100–900 nm typically studied in the spectra of type-Ia supernovae, the energy factor E_i^3 is of order 10^{-24} GeV^3 or smaller, so a value $R\gtrsim1$ would require $n-1\gtrsim10^{10}$, which is well outside the perturbative regime. Moreover, the energy dependence implies that the effect varies by orders of magnitude over an observed spectrum, which for large values of $n-1$ would distort spectra beyond observed limits. The requirement of perturbative $n-1$ evidently restricts substantial redshift effects to high-energy photons. However, it remains conceivable that a detailed analysis along the above lines could extract additional constraints on $s^{(d)}$ from precision cosmological measurements.

In the above, we consider gravitational Čerenkov radiation by photons. However, Lorentz violation in the pure-gravity sector can also cause electromagnetic Čerenkov radiation by gravitons, corresponding to graviton decay. Discussion of this process is lacking in the literature. The form of the Einstein–Maxwell Lagrange density (31) reveals that at leading order this process involves two-photon emission, being governed by a photon-photon-graviton-graviton vertex. The corresponding amplitude is proportional to G_N, and hence on dimensional grounds the power loss of the graviton takes the form

$$\frac{dE}{dt}=-F'(d)G_N^2(n-1)^k|\vec{l}|^6, \tag{52}$$

where $F'(d)$ is a dimensionless factor depending on d arising from integration of the matrix element, k is the power of the dimensionless combination $n-1$ emerging from the matrix element, and $|\vec{l}|$ is the magnitude of the graviton momentum. This result implies that the frequency spectrum of gravity waves detected by a gravitational-wave observatory on or near the Earth is distorted and downshifted by Lorentz violation. However, the effect is far below observational levels in practice, both because the power loss (52) is proportional to G_N^2 and because the energy of typical gravitational waves is expected to be tiny. For example, a gravitational wave in the LIGO band at frequency 100 Hz originating in our galaxy experiences a negligible frequency shift $\delta\nu \approx 10^{-150}(n-1)^k$ Hz. For similar reasons, graviton Čerenkov decay into other particle species is negligible as well.

The results in this work complement those obtained in tests of short-range gravity [19–21] and thereby improve the coverage of sensitivities to coefficients for Lorentz violation in the gravity sector. Exploring the remaining coefficients for even d and the coefficients for odd d, all of which are birefringent, is an interesting open problem for future research.

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References

[1] V.A. Kostelecký, N. Russell, Data tables for Lorentz and CPT violation, 2015 edition, arXiv:0801.0287v8.
[2] V.A. Kostelecký, S. Samuel, Phys. Rev. D 39 (1989) 683;
V.A. Kostelecký, R. Potting, Nucl. Phys. B 359 (1991) 545;
V.A. Kostelecký, R. Potting, Phys. Rev. D 51 (1995) 3923.
[3] V.A. Kostelecký, Phys. Rev. D 69 (2004) 105009.
[4] J.B.R. Battat, J.F. Chandler, C.W. Stubbs, Phys. Rev. Lett. 99 (2007) 241103.
[5] H. Müller, et al., Phys. Rev. Lett. 100 (2008) 031101.
[6] K.-Y. Chung, et al., Phys. Rev. D 80 (2009) 016002.
[7] D. Bennett, V. Skavysh, J. Long, in: V.A. Kostelecký (Ed.), CPT and Lorentz Symmetry V, World Scientific, Singapore, 2011.
[8] L. Iorio, Class. Quantum Gravity 29 (2012) 175007.
[9] Q.G. Bailey, R.D. Everett, J.M. Overduin, Phys. Rev. D 88 (2013) 102001.
[10] L. Shao, Phys. Rev. Lett. 112 (2014) 111103;
L. Shao, Phys. Rev. D 90 (2014) 122009.
[11] A. Hees, Q.G. Bailey, C. Le Poncin-Lafitte, A. Bourgoin, A. Rivoldini, B. Lamine, F. Meynadier, C. Guerlin, P. Wolf, arXiv:1508.03478.
[12] Q.G. Bailey, V.A. Kostelecký, Phys. Rev. D 74 (2006) 045001.
[13] M.D. Seifert, Phys. Rev. D 79 (2009) 124012;
M.D. Seifert, Phys. Rev. D 81 (2010) 065010.
[14] B. Altschul, Q.G. Bailey, V.A. Kostelecký, Phys. Rev. D 81 (2010) 065028.
[15] Q.G. Bailey, R. Tso, Phys. Rev. D 84 (2011) 085025.
[16] J.D. Tasson, Phys. Rev. D 86 (2012) 124021.
[17] Y. Bonder, Phys. Rev. D 91 (2015) 125002.
[18] Q.G. Bailey, V.A. Kostelecký, R. Xu, Phys. Rev. D 91 (2015) 022006.
[19] J.C. Long, V.A. Kostelecký, Phys. Rev. D 91 (2015) 092003.
[20] C.-G. Shao, Y.-J. Tan, W.-H. Tan, S.-Q. Yang, J. Luo, M.E. Tobar, Phys. Rev. D 91 (2015) 102007.
[21] C.-G. Shao, et al., in preparation.
[22] J.D. Tasson, Rep. Progr. Phys. 77 (2014) 062901.
[23] C.M. Will, Living Rev. Relativ. 17 (2014) 4.
[24] R. Bluhm, Lect. Notes Phys. 702 (2006) 191.
[25] P.A. Čerenkov, Dokl. Akad. Nauk SSSR 2 (1934) 451;
S.I. Vavilov, Dokl. Akad. Nauk SSSR 2 (1934) 457.
[26] O. Heaviside, Philos. Mag. 27 (1889) 324;
I.E. Tamm, I.M. Frank, Dokl. Akad. Nauk SSSR 14 (1937) 107;
E. Fermi, Rev. Mod. Phys. 57 (1940) 485.
[27] D. Colladay, V.A. Kostelecký, Phys. Rev. D 55 (1997) 6760;
D. Colladay, V.A. Kostelecký, Phys. Rev. D 58 (1998) 116002.
[28] S.R. Coleman, S.L. Glashow, Phys. Rev. D 59 (1999) l16008.
[29] F.W. Stecker, S.L. Glashow, Astropart. Phys. 16 (2001) 97.
[30] T. Jacobson, S. Liberati, D. Mattingly, Phys. Rev. D 67 (2003) 124011.
[31] O. Gagnon, G.D. Moore, Phys. Rev. D 70 (2004) 065002.
[32] R. Lehnert, R. Potting, Phys. Rev. Lett. 93 (2004) 110402;
R. Lehnert, R. Potting, Phys. Rev. D 70 (2004) 125010.
[33] B. Altschul, Phys. Rev. Lett. 98 (2007) 041603;
B. Altschul, Phys. Rev. D 75 (2007) 105003.
[34] B. Altschul, Astropart. Phys. 28 (2007) 380.
[35] B. Altschul, Nucl. Phys. B 796 (2008) 262.
[36] F.R. Klinkhamer, M. Risse, Phys. Rev. D 77 (2008) 016002.
[37] M. Schreck, arXiv:1310.5159.
[38] M.A. Hohensee, R. Lehnert, D.F. Phillips, R.L. Walsworth, Phys. Rev. Lett. 102 (2009) 170402;
M.A. Hohensee, R. Lehnert, D.F. Phillips, R.L. Walsworth, Phys. Rev. D 80 (2009) 036010.
[39] V.A. Kostelecký, M. Mewes, Phys. Rev. D 85 (2012) 096005;
J.S. Díaz, V.A. Kostelecký, M. Mewes, Phys. Rev. D 89 (2014) 043005.
[40] V.A. Kostelecký, M. Mewes, Phys. Rev. D 88 (2013) 096006.
[41] P.C. Peters, Phys. Rev. D 9 (1974) 2207.
[42] P. Szekeres, Ann. Phys. 64 (1971) 599.
[43] A.G. Polnarev, Sov. Phys. JETP 35 (1972) 834.
[44] D. Chesters, Phys. Rev. D 7 (1973) 2863.
[45] M. Pardy, Phys. Lett. B 336 (1994) 362.
[46] G.D. Moore, A.E. Nelson, J. High Energy Phys. 09 (2001) 023.
[47] J.W. Elliott, G.D. Moore, H. Stoica, J. High Energy Phys. 0508 (2005) 066.
[48] R. Kimura, K. Yamamoto, J. Cosmol. Astropart. Phys. 07 (2012) 050.
[49] M. De Laurentis, S. Capozziello, G. Basini, Mod. Phys. Lett. A 27 (2012) 1250136.
[50] V.A. Kostelecký, M. Mewes, Phys. Rev. D 80 (2009) 015020.
[51] V.A. Kostelecký, M. Mewes, in preparation.
[52] R. Bluhm, V.A. Kostelecký, Phys. Rev. D 71 (2005) 065008;
R. Bluhm, S.-H. Fung, V.A. Kostelecký, Phys. Rev. D 77 (2008) 065020.
[53] N. Globus, D. Allard, E. Parizot, Phys. Rev. D 92 (2015) 021302.
[54] See, for example, A.M. Hillas, Annu. Rev. Astron. Astrophys. 22 (1984) 425;
K.V. Ptitsyna, S.V. Troitsky, Phys. Usp. 53 (2010) 691.
[55] K. Greisen, Phys. Rev. Lett. 16 (1966) 748;
G.T. Zatsepin, V.A. Kuz'min, JETP Lett. 4 (1966) 78.
[56] K. Kotera, A.V. Olinto, Annu. Rev. Astron. Astrophys. 49 (2011) 119.
[57] Catalog of the highest energy cosmic rays, http://eas.ysn.ru/catalog.
[58] M. Takeda, et al., Astrophys. J. 522 (1999) 225, arXiv:astro-ph/9902239.
[59] D.J. Bird, et al., Astrophys. J. 441 (1995) 144.
[60] G. Cunningham, et al., in: M. Wada (Ed.), Catalogue of Highest Energy Cosmic Rays, No. 1, IPCR Itabashi, Tokyo, 1980.
[61] R.U. Abbasi, et al., Astropart. Phys. 30 (2008) 175, http://www.cosmic-ray.org/supplements.html.
[62] A. Aab, et al., Astrophys. J. 804 (2015) 15.
[63] M.M. Winn, J. Ulrichs, L.S. Peak, C.B.A. McCusker, L. Horton, J. Phys. G 12 (1986) 653.
[64] U.R. Abbasi, et al., Astrophys. J. 790 (2014) L21.
[65] J. Linsley, in: M. Wada (Ed.), Catalogue of Highest Energy Cosmic Rays, No. 1, IPCR Itabashi, Tokyo, 1980.
[66] M.I. Pravdin, et al., in: Proc. 29th Int. Cosmic Ray Conf. Pune, vol. 7, 2005, p. 243.
[67] See, for example, G.B. Dantzig, M.N. Thapa, Linear Programming 1: Introduction, Springer, New York, 1997.
[68] J.S. Díaz, V.A. Kostelecký, M. Mewes, Phys. Rev. D 89 (2014) 043005.
[69] R. Bluhm, V.A. Kostelecký, C.D. Lane, N. Russell, Phys. Rev. D 68 (2003) 125008;
R. Bluhm, V.A. Kostelecký, C.D. Lane, N. Russell, Phys. Rev. Lett. 88 (2002) 090801;
V.A. Kostelecký, M. Mewes, Phys. Rev. D 66 (2002) 056005.
[70] V.A. Kostelecký, Phys. Lett. B 701 (2011) 137.
[71] J. Foster, R. Lehnert, Phys. Lett. B 746 (2015) 164;
N. Russell, Phys. Rev. D 91 (2015) 045008;
M. Schreck, Phys. Rev. D 91 (2015) 105001;
M. Schreck, Eur. J. Phys. C 75 (2015) 187;
M. Schreck, arXiv:1508.00216;
J.E.G. Silva, C.A.S. Almeida, Phys. Lett. B 731 (2014) 74;
V.A. Kostelecký, N. Russell, R. Tso, Phys. Lett. B 716 (2012) 470;
D. Colladay, P. McDonald, Phys. Rev. D 85 (2012) 044042;
D. Colladay, P. McDonald, arXiv:1507.01018.
[72] V.A. Kostelecký, N. Russell, Phys. Lett. B 693 (2010) 443;
V.A. Kostelecký, R. Lehnert, Phys. Rev. D 63 (2001) 065008.
[73] V.A. Kostelecký, J.D. Tasson, Phys. Rev. Lett. 102 (2009) 010402;
V.A. Kostelecký, J.D. Tasson, Phys. Rev. D 83 (2011) 016013.
[74] V.A. Kostelecký, R.J. Van Kooten, Phys. Rev. D 82 (2010) 101702 (R);
V.A. Kostelecký, Phys. Rev. Lett. 80 (1998) 1818.
[75] LIGO Collaboration, J. Aasi, et al., Class. Quantum Gravity 32 (2015) 074001.
[76] Virgo Collaboration, F. Acernese, et al., Class. Quantum Gravity 32 (2015) 024001.
[77] eLISA Consortium, P. Amaro Seoane, et al., arXiv:1305.5720.
[78] D. Hansen, N. Yunes, K. Yagi, Phys. Rev. D 91 (2015) 082003.
[79] V.A. Kostelecký, A.G.M. Pickering, Phys. Rev. Lett. 91 (2003) 031801.
[80] F. Zwicky, Proc. Natl. Acad. Sci. 15 (1929) 773;
O. Halpern, Phys. Rev. 44 (1933) 855;
P. Roman, Magy. Fiz. F. 3 (1955) 115;
D.C. Wilkins, Phys. Rev. D 21 (1980) 2122.
[81] S. Blondin, et al., Ap. J. 682 (2008) 724.
[82] See, for example, S. Weinberg, Cosmology, Oxford University Press, Oxford, England, 2008.